Vectors

Questions?

Position in 1D

Position in 2D

- Transforming between coordinate systems
- Arctan subtlety
- Addition
- Scalar Product

Position in 3D

- Extend definitions
- Cross product

To specify <u>position</u> in <u>one</u> dimension, you need one number (with a sign).

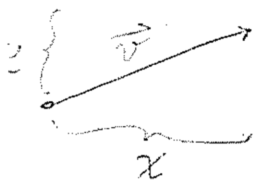
Generic Problem Setup:

- Choose a location to call $x = 0$.
- Refer to other locations by $x =$ (displacement from 0).
- Add or subtract positions to solve problems.

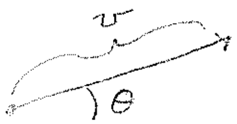
Example:

I bike directly south on Charles from campus to the Harbor 3 miles away. When I am one mile along, how far am I from the Harbor?

To specify position in <u>two</u> dimensions, we need to know two numbers.



Cartesian ← Notice these require a <u>sign</u>...
"over, up"



magnitude ← ... but these are perfectly
& direction unambiguous without one.

<u>Notation</u>

$$\vec{v} = (v_x, v_y) \qquad (5, 3)$$

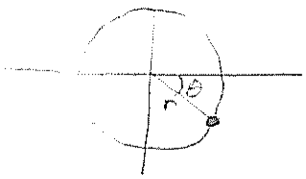
$$= v_x\hat{x} + v_y\hat{y} \qquad 5\hat{x} + 3\hat{y}$$

$$= v_x\hat{i} + v_y\hat{j} \qquad 5\hat{i} + 3\hat{j}$$

Transforming Between Coordinates

Capt Cook sails x miles east and y miles south. How far has he gone, and in which direction?

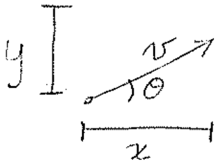
Pythagorean thrm: $r = \sqrt{x^2 + y^2}$



$\tan\theta = \frac{y}{x}$

$\theta = \arctan \frac{y}{x}$

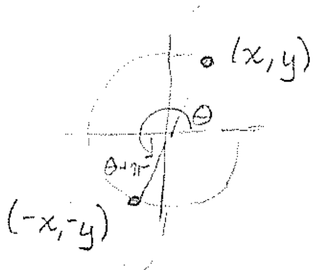
Capt. Cook sails r miles θ degress north of due east. How far east has he travelled? How far north?



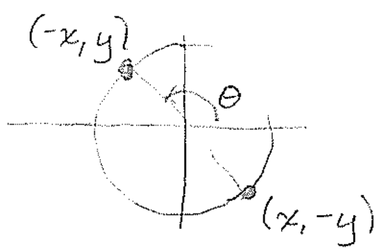
$x = r\cos\theta$
$y = r\sin\theta$

Arctangent Subtlety

$\tan\theta = \tan(\theta + 180°)$



$\tan\theta = \frac{y}{x}$
$\tan(\theta + 180°) = \frac{-y}{-x} = \tan\theta$



$\tan\theta = \frac{y}{-x}$
$\tan(\theta + 180°) = \frac{-y}{x} = \tan\theta$

Quadrant	Question	Calculator Answer	True Answer
I	$\arctan \frac{y}{x}$	θ	θ
III	$\arctan \frac{-y}{-x}$	θ	$\theta + 180°$
II	$\arctan \frac{y}{-x}$	θ	θ
IV	$\arctan \frac{-y}{x}$	θ	$\theta + 180°$

Moral: Draw a picture!

Capt. Cook ends up 3 miles south, 3 miles west of his starting point. In what direction did he sail?

$x = -3$

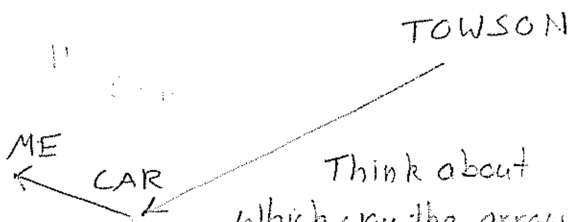
$y = -3$

$\theta = \arctan \frac{y}{x} = 45°$ or $45° + 180°$ $= 135°$

I am 2 mi. W, 1 mi. N of my car, which is parked 6 mi. S, 8 mi. W of Towson. Where is Towson relative to me? We need to add vectors.

$\vec{a} = -2\hat{x} + 1\hat{y}$

$\vec{b} = -8\hat{x} - 6\hat{y}$



Think about which way the arrows point.

$\vec{a} + \vec{b} = (-2-8)\hat{x} + (1-6)\hat{y} = -10\hat{x} - 5\hat{y}$

Me relative to Towson

$\vec{v} = -(\vec{a} + \vec{b}) = 10\hat{x} + 5\hat{y}$

Towson relative to me

→

How far away is that, directly?

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{125} \approx 11 \text{ mi.}$$

In what direction?

$$\theta = \arctan \frac{v_y}{v_x} = \text{in quadrant I}$$

Many problems go like this:

GIVEN several steps with magnitude & direction → [CONVERT to component form → ADD component-wise → CONVERT result back to mag. & dir.] → ANSWER total magnitude & direction after all steps

Or find the displacement between the final position and some other position.

$$\vec{d} = \vec{o} - \vec{f}$$

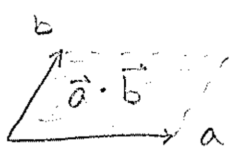
Two kinds of product:

① Dot Product, Scalar Product, Inner Product

Result is a scalar.

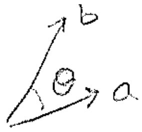
Projection of one vector along another.

Geometric interpretation in 2D:



Equivalent Def:

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$ — Know components?

$\vec{a} \cdot \vec{b} = a b \cos\theta$, angle between 

Lengths?

* Notice that the self scalar product is the magnitude.

Example: Given components, find angle between.

$\vec{a} = (4, 3)$

$\vec{b} = (7, 7)$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y = 28 + 21 = 49$

$a = \sqrt{a_x^2 + a_y^2} = 5$

$b = 7\sqrt{2}$

$\vec{a} \cdot \vec{b} = ab\cos\theta = 35\sqrt{2}\cos\theta$

$49 = 35\sqrt{2}\cos\theta$

$\theta = \arccos\left(\frac{49}{35\sqrt{2}}\right)$

To specify position in three dimensions, we need 3 numbers. Extend definitions easily.

$\vec{v} = (v_x, v_y, v_z)$

$= v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$

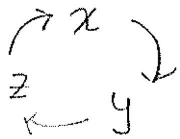
$= v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Cross Product — result is a <u>vector</u>

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$



cyclical

$\vec{a}\times\vec{b} = ab\sin\theta$; in a direction normal to the plane of $\vec{a}$ and $\vec{b}$.

A vector product contains <u>3 equations</u> that are separately, simultaneously satisfied.

$$\vec{v} = \vec{a}\times\vec{b} \implies$$

$$\begin{cases} v_x = a_y b_z - a_z b_y \\ v_y = a_z b_x - a_x b_z \\ v_z = a_x b_y - a_y b_x \end{cases}$$

Example:

You know the components of $\vec{v}$ and $\vec{a}$. What are the components of $\vec{b}$?

If these were scalar numbers, it'd be easy: $v = ab \Rightarrow b = \frac{v}{a}$.

We have 3 equations, 3 unknown variables. Could solve with algebra. (Ex: $\vec{v} = (-3,6,-3)$, $\vec{a} = (1,2,3) \Rightarrow \vec{b} = (4,5,6)$.)